Shapley-Shubik Power Index in Weighted Double Majority Games: A Review

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ABSTRACT: The Shapley-Shubik power index in a voting situation depends on the number of orderings in each player is pivotal. Shapley-Shubik power index is one of the voting power indexes which measure the strength of every player in a voting game. The main purpose of this work is to review the computation of Shapley-Shubik power in weighted double majority games through generating functions.

Key Words: Power Index; Power Indices; Shapley-Shubik; Voting rule

Introduction

The main objective of power indices is to evaluate the a priori influence of a given player for a given voting rule by computing the number of times he or she is decisive. Most of the literature on power indices were devoted to the evaluation of the influence of the different players in different perspective e.g. United Nation, US state electoral college, Political parties, Shareholders.

Shapley-Shubik power index is among the most widespread power indices. The Shapley-Shubik power index was introduced in 1950’(Shapley and Shubik 1954). The index is based on the Shapley value introduced by Lloyd Shapley (Shapley 1953). Shapley-Shubik (1954) introduced a voting scheme as well as the idea of the pivotal or decisive player. This article reviews the computation of Shapley-Shubik power index through generating functions.

Definition:
A cooperative game is a function $T: 2^N \rightarrow i$ such that $T(\emptyset) = 0$, where $N = \{1, 2, \ldots, n\}$ is the collection of players or voters.

Definition:
A simple game is a cooperative game $T: 2^N \rightarrow \{0, 1\}$ such that $T(N) = 1$ and $T(S) \subseteq T(M)$ whenever $S \subseteq M \subseteq N$. Denote simple game by $(N, T)$

Definition:
A weighted voting game denoted by $[q; w_1, w_2, \ldots, w_n]$ is a class of a simple game where $q$ is the quota and $w_i$, $i \in N$ is the weight attached to player $i$
Such that $w_n \leq w_{n-1} \leq \ldots \leq w_1$ and $w_i \leq q \leq \sum_{j=1}^{n} w_j$.

A weighted double majority game is the simple game $\big( N, T_1 \land T_2 \big)$ where $(N, T_t)$ are the weighted voting games denoted by $\big[ q^t; w_1^t, \ldots, w_n^t \big]$ for $1 \leq t \leq 2$ whose characteristic function is denoted by

$$(T_1 \land T_2)(S) = \begin{cases} 1 & \text{for if } w'(S) \geq q' \\ 0, & \text{otherwise} \end{cases}$$

where $w'(S) = \sum_{i \in S} w_i$.

### Procedure

The Shapley–Shubik power index for player $i \in N$ can be defined by

$$T_i(v) = \sum_{\{S \in W : S \cup \{i\} \in W\}} \frac{s!(n-s-1)!}{n!}$$

where $n = |N|, s = |S|$ and $W$ is the set of Winning coalition.

The Shapley–Shubik power index distribution is the vector

$$T(v) = (T_1(v), \ldots, T_n(v))$$

where $T_i(v)$ is defined as above. The value $T_i(v)$ can also be define by

$$T_i(v) = \sum_{j=0}^{n-1} \frac{j!(N-j-1)!}{N!} \eta_j^i,$$

where each

$\eta_j^i$ is the number of swings of player $i$ in coalition of size $j$.

David G. Cantor apply generating function to compute the index for weighted voting games. His contribution culminated to the following:

For a weighted voting game $v = [q; w_1, \ldots, w_n]$ the Shapley–Shubik index of player $i$ satisfies

$$T_i(v) = \sum_{j=0}^{n-1} \frac{j!(N-j-1)!}{n!} \left( \frac{\sum_{k=q \sim w_j} a_{kj}^i}{\sum_{k=q \sim w_j} a_{kj}^i} \right)$$

where $a_{kj}^i$ is the number of coalition $S$ of $j$ players with $w(S) = k$ such that $i \not\in S$.

Thus,
Proposition

Let \((N, v)\) be a weighted double majority game with \(v = v_1 \land v_2\) such that
\[v_i = \begin{bmatrix} q^i; w_1^i, w_2^i, \ldots, w_n^i \end{bmatrix}, 1 \leq i \leq 2.\]
Then for every \(i \in N\), the generating functions of the numbers \(\{a_{k_1-k_m}^i\}_{k_1-k_m, j=0}\) are given by
\[S_i(x, y, z) = \prod_{j=1}^{n} (1 + x^{w_j^i} y^{p_j^i} z)\]

For the weighted double majority game, \((N, v)\) with \(v = v_1 \land v_2\) where
\[v_1 = \begin{bmatrix} q; w_1, w_2, \ldots, w_n \end{bmatrix}\] and \(v_2 = \begin{bmatrix} t; p_1, p_2, \ldots, p_n \end{bmatrix}\), the generating function of \(\{a_{k_r^i}^j\}_{k_r^i, j=0}\) where \(a_{k_r^i}^j\) is the number of coalitions \(S\) of \(j\) players with \(w(S) = k, p(S) = r\) and such that \(i \not\in S\), is given by
\[S_i(x, y, z) = \prod_{j=1}^{n} (1 + x^{w_j^i} y^{p_j^i} z)\]

The number of swings of player \(i\) in the coalition of size \(j\) is given by
\[\eta_j^i = \sum_{k=q-w_1}^{w(N^i)} \sum_{r=l-p_1}^{w(N^i)} a_{k_r^i}^j - \sum_{k=q}^{w(N^i)} \sum_{r=l}^{w(N^i)} a_{k_r^j}^j,\]
where \(a_{k_r^j}^j\) is the number of coalitions \(S\) such that \(i \not\in S\) with \(w(S) = k, p(S) = r\) and \(|S| = j\).

Example

Consider the weighted double majority game \(v = v_1 \land v_2\), where \(v_1 = \begin{bmatrix} 8; 5, 3, 2, 2 \end{bmatrix}\) and \(v_2 = \begin{bmatrix} 3, 1, 1, 1, 1 \end{bmatrix}\), whose characteristic function is given by
\[v_1 \land v_2(S) = \begin{cases} 1, & \text{if } w^1(S) \geq 8 \text{ and } w^2(S) \geq 3 \\ 0, & \text{otherwise} \end{cases}\]
we apply the function
\[S_i(x, y, z) = \prod_{j=1}^{n} (1 + x^{w_j^i} y^{p_j^i} z)\]
\[ S_1(x, y, z) = 1 + 2x^2yz + x^3yz + x^4y^2z^2 + 2x^5y^2z^2 + x^7y^3z^3. \]

\[ S_2(x, y, z) = 1 + 2x^2yz + x^4y^2z^2 + x^5yz + 2x^7y^2z^2 + x^9y^3z^3 \]

\[ S_3(x, y, z) = (1 + x^5yz)(1 + x^3yz)(1 + x^2yz) = (1 + x^5yz)[1 + x^2yz + x^3yz + x^5y^2z^2] = 1 + x^2yz + x^3yz + x^5y^2z^2 + x^7y^2z^2 + x^8y^2z^2 + x^{10}y^3z^3 \]

\[ S_4(x, y, z) = (1 + x^3yz)(1 + x^3yz)(1 + x^2yz) = (1 + x^5yz)[1 + x^2yz + x^3yz + x^5y^2z^2] = 1 + x^2yz + x^3yz + x^5y^2z^2 + x^7y^2z^2 + x^8y^2z^2 + x^{10}y^3z^3. \]

To compute the Shapley–Shubik index for each player, we use

\[ T_i(v) = \sum_{j=0}^{n-1} \frac{j!(n-j-1)!}{n!} \eta_j^i \]

\[ \sum_{j=0}^{n-1} \frac{j!(n-j-1)!}{n!} \left( \sum_{k=q-w_{ij}}^{r-\eta_j^i} \sum_{l=r-p_i} \sum_{k=q} \sum_{r=l} a_{krj}^i - \sum_{k=q} \sum_{r=l} a_{krj}^i \right) = \sum_{j=0}^{n-1} \frac{j!(n-j-1)!}{n!} \left( \sum_{k=q} \sum_{r=l} a_{krj}^i \right) = \sum_{j=0}^{n-1} \frac{j!(n-j-1)!}{n!} \left( a_{1320} + a_{1330} + a_{1420} + a_{1430} + a_{2320} + a_{2330} + a_{2420} + a_{2430} + a_{3220} + a_{3230} + a_{3420} + a_{3430} + a_{4220} + a_{4230} + a_{4320} + a_{4330} + a_{5220} + a_{5230} + a_{5320} + a_{5330} + a_{6220} + a_{6230} + a_{6320} + a_{6330} + a_{7220} + a_{7230} + a_{7320} + a_{7330} \right) + \frac{2!}{4!} \left( a_{1321} + a_{1331} + a_{1421} + a_{1431} + a_{2321} + a_{2331} + a_{2421} + a_{2431} + a_{3221} + a_{3231} + a_{3421} + a_{3431} + a_{4221} + a_{4231} + a_{4321} + a_{4331} + a_{5221} + a_{5231} + a_{5321} + a_{5331} + a_{6221} + a_{6231} + a_{6321} + a_{6331} + a_{7221} + a_{7231} + a_{7321} + a_{7331} \right) + \frac{2!}{4!} \left( a_{1322} + a_{1332} + a_{1422} + a_{1432} + a_{2322} + a_{2332} + a_{2422} + a_{2432} + a_{3222} + a_{3232} + a_{3422} + a_{3432} + a_{4222} + a_{4232} + a_{4322} + a_{4332} + a_{5222} + a_{5232} + a_{5322} + a_{5332} + a_{6222} + a_{6232} + a_{6322} + a_{6332} + a_{7222} + a_{7232} + a_{7322} + a_{7332} \right) + \frac{3!}{4!} \left( a_{1323} + a_{1333} + a_{1423} + a_{1433} + a_{2323} + a_{2333} + a_{2423} + a_{2433} + a_{3223} + a_{3233} + a_{3423} + a_{3433} + a_{4223} + a_{4233} + a_{4323} + a_{4333} + a_{5223} + a_{5233} + a_{5323} + a_{5333} + a_{6223} + a_{6233} + a_{6323} + a_{6333} + a_{7223} + a_{7233} + a_{7323} + a_{7333} \right) . \]

\[ \frac{3!}{4!} (0) + \frac{2!}{4!} (0) + \frac{2!}{4!} (1+2) + \frac{3!}{4!} (1) = \frac{6}{24} + \frac{6}{24} = \frac{1}{2} \]

\[ T_2(v) = \sum_{j=0}^{n-1} \frac{j!(n-j-1)!}{n!} \left( \sum_{k=5}^{3} \sum_{r=2}^{3} a_{krj}^2 - \sum_{k=8}^{3} \sum_{r=3}^{3} a_{krj}^2 \right) = }
\[
\frac{3}{n!} \sum_{j=0}^{3} j!(n-j-1)! (\sum_{k=6}^{9} a_{k2j}^2 + a_{k3j}^2 - \sum_{k=8}^{9} a_{k3j}^2) = \\
\frac{3}{n!} \sum_{j=0}^{3} j!(n-j-1)! \left( a_{52j}^2 + a_{53j}^2 + a_{62j}^2 + a_{63j}^2 + a_{72j}^2 + a_{73j}^2 + a_{82j}^2 + a_{82j}^2 \right)
\]

\[
\frac{0!3!}{4!} \left( a_{520}^2 + a_{530}^2 + a_{620}^2 + a_{630}^2 + a_{720}^2 + a_{730}^2 + a_{820}^2 + a_{820}^2 \right)
\]

\[
\frac{1!2!}{4!} \left( a_{621}^2 + a_{631}^2 + a_{621}^2 + a_{631}^2 + a_{721}^2 + a_{731}^2 + a_{821}^2 + a_{821}^2 \right)
\]

\[
\frac{2!1!}{4!} \left( a_{522}^2 + a_{532}^2 + a_{622}^2 + a_{632}^2 + a_{722}^2 + a_{732}^2 + a_{822}^2 + a_{822}^2 \right)
\]

\[
\frac{3!0!}{4!} \left( a_{523}^2 + a_{533}^2 + a_{623}^2 + a_{633}^2 + a_{723}^2 + a_{733}^2 + a_{823}^2 + a_{823}^2 \right)
\]

\[
\frac{0!3!}{4!} \left( 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 \right) + \frac{1!2!}{4!} \left( 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 \right)
\]

\[
\frac{2!1!}{4!} \left( 0 + 0 + 0 + 0 + 2 + 0 + 0 + 0 \right) + \frac{3!0!}{4!} \left( 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 \right) = \frac{1}{6}
\]

\[
T_3(v) = \frac{3}{n!} \sum_{j=0}^{3} j!(n-j-1)! \left( \sum_{k=6}^{10} a_{krj}^3 - \sum_{r=3}^{10} a_{krj}^3 \right) = \\
\frac{3}{n!} \sum_{j=0}^{3} j!(n-j-1)! \left( \sum_{k=6}^{10} a_{k2j}^3 + a_{k3j}^3 - \sum_{k=8}^{10} a_{k3j}^3 \right) = \\
\frac{3}{n!} \sum_{j=0}^{3} j!(n-j-1)! \left( a_{62j}^3 + a_{63j}^3 + a_{72j}^3 + a_{73j}^3 + a_{82j}^3 + a_{83j}^3 + a_{92j}^3 + a_{93j}^3 + a_{102j}^3 + a_{103j}^3 \right)
\]

\[
\frac{0!3!}{4!} \left( a_{620}^3 + a_{630}^3 + a_{720}^3 + a_{820}^3 + a_{920}^3 + a_{1020}^3 \right)
\]

\[
\frac{1!2!}{4!} \left( a_{621}^3 + a_{631}^3 + a_{721}^3 + a_{821}^3 + a_{921}^3 + a_{1021}^3 \right)
\]

\[
\frac{2!1!}{4!} \left( a_{622}^3 + a_{623}^3 + a_{722}^3 + a_{723}^3 + a_{822}^3 + a_{823}^3 + a_{922}^3 + a_{1022}^3 \right)
\]

\[
T_3(v) = \sum_{j=0}^{3} j!(n-j-1)! \left( \sum_{k=6}^{10} a_{krj}^3 - \sum_{r=3}^{10} a_{krj}^3 \right) = \\
\sum_{j=0}^{3} j!(n-j-1)! \left( \sum_{k=6}^{10} a_{k2j}^3 + a_{k3j}^3 - \sum_{k=8}^{10} a_{k3j}^3 \right) = \\
\sum_{j=0}^{3} j!(n-j-1)! \left( a_{62j}^3 + a_{63j}^3 + a_{72j}^3 + a_{73j}^3 + a_{82j}^3 + a_{83j}^3 + a_{92j}^3 + a_{93j}^3 + a_{102j}^3 + a_{103j}^3 \right)
\]

\[
\frac{0!3!}{4!} \left( a_{620}^3 + a_{630}^3 + a_{720}^3 + a_{820}^3 + a_{920}^3 + a_{1020}^3 \right) + \\
\frac{1!2!}{4!} \left( a_{621}^3 + a_{631}^3 + a_{721}^3 + a_{821}^3 + a_{921}^3 + a_{1021}^3 \right) + \\
\frac{2!1!}{4!} \left( a_{622}^3 + a_{623}^3 + a_{722}^3 + a_{723}^3 + a_{822}^3 + a_{823}^3 + a_{922}^3 + a_{1022}^3 \right)
\]

\[
T_3(v) = \sum_{j=0}^{3} j!(n-j-1)! \left( \sum_{k=6}^{10} a_{krj}^3 - \sum_{r=3}^{10} a_{krj}^3 \right) = \\
\sum_{j=0}^{3} j!(n-j-1)! \left( \sum_{k=6}^{10} a_{k2j}^3 + a_{k3j}^3 - \sum_{k=8}^{10} a_{k3j}^3 \right) = \\
\sum_{j=0}^{3} j!(n-j-1)! \left( a_{62j}^3 + a_{63j}^3 + a_{72j}^3 + a_{73j}^3 + a_{82j}^3 + a_{83j}^3 + a_{92j}^3 + a_{93j}^3 + a_{102j}^3 + a_{103j}^3 \right)
\]

\[
\frac{0!3!}{4!} \left( a_{620}^3 + a_{630}^3 + a_{720}^3 + a_{820}^3 + a_{920}^3 + a_{1020}^3 \right) + \\
\frac{1!2!}{4!} \left( a_{621}^3 + a_{631}^3 + a_{721}^3 + a_{821}^3 + a_{921}^3 + a_{1021}^3 \right) + \\
\frac{2!1!}{4!} \left( a_{622}^3 + a_{623}^3 + a_{722}^3 + a_{723}^3 + a_{822}^3 + a_{823}^3 + a_{922}^3 + a_{1022}^3 \right)
\]

\[
185
\]
The Shapley – Shubik power index distribution is the vector \( T(v) = (T_1(v), T_2(v), T_3(v), T_4(v)) = \left( \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6} \right) \) which indicate the distribution of power or strength of each player or Member.
Conclusion

This paper computes the Shapley-Shubik power index by using generating functions for weighted double majority games. The paper discusses the computational method of Shapley-Shubik power index in double majority games. It also obtains the results and the distribution of power among the players.

The Shapley-Shubik power index reflects the power and the actual contributions of players to the formation of winning coalitions in decision making.

References


